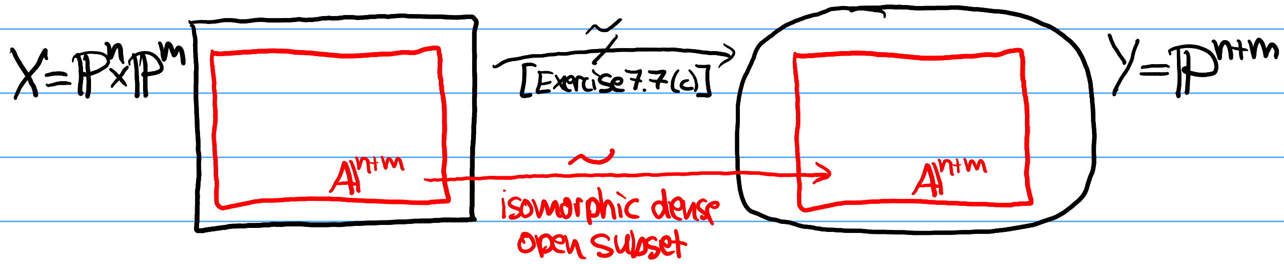
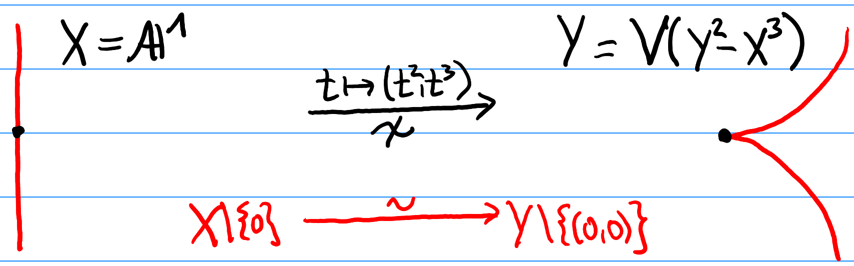
9. Birational maps and Blowing Up

Big picture

- Have collected many examples of varieties (affine, projective, Grassmannians) and construction techniques of new ones (gluing, products)
- Some pairs of these examples are "almost the same":





- in this chapter
  - ⇝ develop language to describe having isom. open subsets
  - ⇝ show construction (blow-up) that modifies an alg. variety X locally (leaving open subset unchanged)

Def (Rational maps)
Let X, Y be irreducible varieties. A rational map $f : X \dashrightarrow Y$ is a morphism $f : U \to Y$ with $\emptyset \neq U \subseteq X$ open.
Equivalence $f_1, f_2 : X \dashrightarrow Y$ defined on U_1, U_2 are equivalent if $\exists\, \emptyset \neq U \subseteq U_1 \cap U_2 : f_1|_U = f_2|_U$.

$$\rightsquigarrow \mathrm{RatMaps}(X,Y) = \left\{ f : \underset{\substack{\cap\!\!| \\ X}}{U} \longrightarrow Y \right\} \Big/ \sim$$

↖ equivalence relation (uses: $\emptyset \neq U \subseteq X$ dense)

Exq. $f: \underbrace{\mathbb{A}^1 \setminus \{0\}}_{U} \to \mathbb{A}^1,\ x \mapsto 1/x$ defines $f: \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$.

- Morphism

$$\underbrace{\mathbb{A}^1 \setminus \{0,1\}}_{U'} \to \mathbb{A}^1,\ x \mapsto 1/x \quad \text{represents same rat'l map}$$

Rmks

(a) $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ agree on $\emptyset \neq U \subseteq U_1 \cap U_2$

$\Longrightarrow f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$ since $\begin{cases} U \text{ dense in } U_1 \cap U_2 \\ \text{locus where } f_1 = f_2 \text{ is closed in } U_1 \cap U_2 \text{ [Pro 5.20(b)]} \end{cases}$

(b) Category of irred. varieties & rational morphisms?

Problem

$f: \mathbb{A}^1 \to \mathbb{A}^1,\ x \mapsto 0$ and $g: \mathbb{A}^1 \dashrightarrow \mathbb{A}^1,\ x \mapsto 1/x$

$\rightsquigarrow$ No good composition map $g \circ f : \mathbb{A}^1 \dashrightarrow \mathbb{A}^1$.

Def (Birational maps) X, Y irred. varieties

(a) $f: X \dashrightarrow Y$ dominant if the image of f is dense in Y

$\rightsquigarrow$ for $\underbrace{g: Y \dashrightarrow Z}_{\text{defined on } V \subseteq Y}$ we get $\underbrace{g \circ f : X \dashrightarrow Z}_{\text{defined on } f^{-1}(V) \neq \emptyset}$

(b) $f: X \dashrightarrow Y$ birational if f is dominant and $\exists\, g: Y \dashrightarrow X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$

as rat'l maps!

(c) X and Y are birational if $\exists$ birational map $f: X \dashrightarrow Y$.

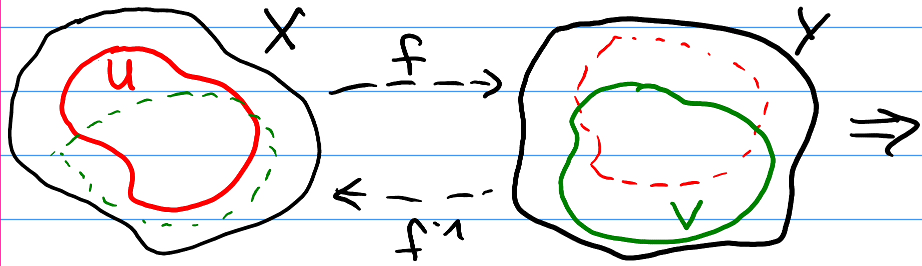
Rmk (for category fans)

Category $\mathrm{Var}_k^{\mathrm{dom}}$: objects = irred. varieties, morph. = dominant rat'l maps $\Big\}$ birat'l maps = isomorphisms.

Started with motivation of identifying X, Y if they contain isomorphic non-empty open subsets.

Lem X, Y irred. varieties. Then
X and Y are birational $\Longleftrightarrow$ X, Y contain isomorphic non-empty open subsets $X \supseteq U \xrightarrow[f]{\sim} V \subseteq Y$.

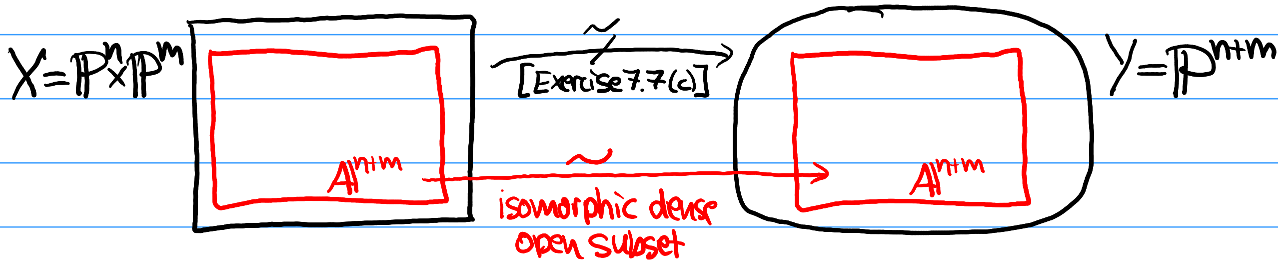
Pf "$\Leftarrow$" $f: X \dashrightarrow Y$ dominant ($\overline{V} = Y$) with inverse $f^{-1}: Y \dashrightarrow X$.
"$\Rightarrow$" $f: X \dashrightarrow Y$ birational, defined on $U \subseteq X$
$g: Y \dashrightarrow X$ inverse, defined on $V \subseteq Y$

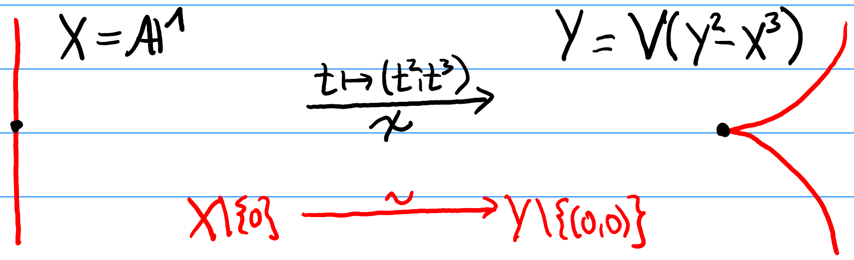


$\Rightarrow$ $f^{-1}(g^{-1}(U)) \subseteq X$, $f \downarrow \quad \uparrow g$ → inverse isom. , $g^{-1}(f^{-1}(V)) \subseteq Y$ $\square$

[Exerc. 5.24] $\Longrightarrow$ birat'l irred. varieties have same dimension.

Exa





Rational functions

Def (Rational function) X irred. variety
A rational map $\varphi: X \dashrightarrow \mathbb{A}^1 = K$ is called a
rational function.

Explicitly rat'l fct.: $\emptyset \neq U \subseteq X$ open, $\varphi \in \mathcal{O}_X(U)$ regular
A pair (U', φ') is equivalent if $\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}$

Construction (Function field) X irred. variety
$K(X) = \{\varphi : X \dashrightarrow \mathbb{A}^1\}$ set of rat'l functions
Claim $K(X)$ is a field.
Pf $\varphi_1 \in \mathcal{O}_X(U_1)$, $\varphi_2 \in \mathcal{O}_X(U_2) \rightsquigarrow \underset{\text{addition}}{\varphi_1 + \varphi_2}$, $\underset{\text{multiplication}}{\varphi_1 \cdot \varphi_2} \in \mathcal{O}_X(\overbrace{U_1 \cap U_2}^{\neq \emptyset})$
$\rightsquigarrow \underset{\text{additive inverse}}{-\varphi_1} \in \mathcal{O}_X(U_1)$

$\rightsquigarrow \varphi \in \mathcal{O}_X(U)$ nonzero in $K(X) \overset{\text{Def}}{\Longrightarrow} U_0 = U \setminus V(\varphi)$ non-empty
$\Longrightarrow \underset{\text{multiplicative inverse}}{(U_0, \varphi^{-1})} = (U, \varphi)^{-1}$ in $K(X)$

$K(X)$: function field of X

Rmks X irred. variety
(a) $U \subseteq X$ non-empty open subvariety
$\Rightarrow \quad K(U) \xrightarrow{\sim} K(X)$
$(\varphi \in \mathcal{O}_X(V)) \longmapsto (\varphi \in \mathcal{O}_X(V))$ $\quad V \subseteq U$ open
$(\varphi|_{U \cap W} \in \mathcal{O}_X(U \cap W)) \longleftarrow (\varphi \in \mathcal{O}_X(W))$ $\quad W \subseteq X$ open

(b) $K(X) =$ stalk of $\mathcal{O}_X$ at X [Exercise 3.23]
$\Rightarrow$ For $U \subseteq X$ open affine : $K(X) = \underset{\text{localiz. at prime ideal } \{0\}}{A(U)_{\langle 0 \rangle}} = \mathrm{Frac}(A(U))$

Fact The contravariant functor $\mathrm{Var}_K^{\mathrm{dom}} \to \mathrm{Fin.Gen.Field\,Extensions}_K$
is an equiv. of categories. $\quad X \mapsto K(X)$

Blow-ups at vanishing loci of functions

Given X variety, describe particular birat'l map $\tilde{X} \to X$

Plan $\begin{pmatrix} X \text{ affine, blow-up} \\ Y = V(\underbrace{f_1, \dots, f_r}_{\text{only dep. on } J = \langle f_1, \dots, f_r \rangle}) \subseteq X \end{pmatrix} \overset{\text{glue}}{\rightsquigarrow} \begin{pmatrix} X \text{ variety, blow-up} \\ \text{at } Y \subseteq X \text{ closed subvar.} \end{pmatrix}$

Construction (Blowing up)

$X \subseteq \mathbb{A}^n$ affine variety, $f_1, \dots, f_r \in A(X)$ Check: morphism!

$\rightsquigarrow f : U = X \setminus V_X(f_1, \dots, f_r) \longrightarrow \mathbb{P}^{r-1}, \; x \mapsto (f_1(x) : \dots : f_r(x))$

$\rightsquigarrow \Gamma_f = \{(x, f(x)) : x \in U\} \overset{\text{closed}}{\subseteq} U \times \mathbb{P}^{r-1} \overset{\text{open}}{\subseteq} X \times \mathbb{P}^{r-1}$

Define $\tilde{X} = \overline{\Gamma_f} \xrightarrow{\pi} X$ the **blow-up** of X at $f_1, \dots, f_r$ ← closure in $X \times \mathbb{P}^{r-1}$

$=: Bl_{f_1, \dots, f_r} X$

Rmks

(a) $\pi|_{\Gamma_f} : \Gamma_f \xrightarrow{\sim} U$ isomorphism $\rightsquigarrow$ see

$$\begin{array}{ccc} U & \overset{\text{open}}{\subseteq} & \tilde{X} \text{ (irreducible)} \\ \downarrow & & \downarrow \pi \\ U & \subseteq & X \end{array}$$

X irreducible, f_i not all zero in $A(X)$ $\rightsquigarrow$ $U \subseteq X$ open, dense

$\Rightarrow \pi$ birational map.

$E = \tilde{X} \setminus U \overset{\text{closed}}{\subseteq} \tilde{X}$ **exceptional set** of blow-up.

(b) $Y \subseteq X$ closed subvariety $\rightsquigarrow$ via restriction: $f_1, \dots, f_r \in A(Y)$

$\Rightarrow \tilde{Y} = Bl_{f_1, \dots, f_r} Y \subseteq Y \times \mathbb{P}^{r-1} \subseteq X \times \mathbb{P}^{r-1}$ closed subvar. of $\tilde{X}$

$=$ closure of $U \cap Y \subseteq U \subseteq \tilde{X}$ in $\tilde{X}$.

$\tilde{Y}$: **strict transform** of Y in $\tilde{X}$

$\underbrace{X = X_1 \cup \dots \cup X_m}_{\text{irred. decompos.}} \rightsquigarrow \tilde{X} = \tilde{X}_1 \cup \dots \cup \tilde{X}_m$ ← blow up irred. components separately

Examples of blow-ups

Exa (trivial examples: $r=1$)
For $r=1$: $\tilde{X} \subseteq X \times \mathbb{P}^0 \cong X$ and $\Gamma_f = U = X \setminus V(f_1)$
$\Rightarrow \tilde{X} = \overline{U} \subseteq X$ closure of complement of $V(f_1)$
If X irreducible:
(a) $f_1 \neq 0 \rightsquigarrow U \subseteq X$ non-empty open $\Rightarrow \tilde{X} = \overline{U} = X$
(b) $f_1 = 0 \rightsquigarrow U = \emptyset \Rightarrow \tilde{X} = \emptyset$.

Goal Explicit description of blow-up (no closures)

Lem X affine, $f_1, \dots, f_r \in A(X)$

$$\Rightarrow \tilde{X} = Bl_{f_1,\dots,f_r} X \subseteq \{(x,y) \in X \times \mathbb{P}^{r-1} : y_i f_j(x) \overset{(*)}{=} y_j f_i(x) \ \forall i,j \in \{1,\dots,r\}\}$$

closed subvariety of $X \times \mathbb{P}^{r-1}$

Pf $U = X \setminus V(f_1, \dots, f_r) \rightsquigarrow$ all $(x,y) \in \Gamma_f \subseteq U \times \mathbb{P}^{r-1}$ satisfy
$(y_1 : \dots : y_r) = (f_1(x) : \dots : f_r(x)) \in \mathbb{P}^{r-1} \iff$ equations $(*)$
$\rightsquigarrow$ equations $(*)$ still hold on closure of $U = \tilde{X}$. $\square$
<u>Note</u> For formal proof: check on affine patches $X \times U_i$, $i = 1, \dots, r$.

Exa (Blow-up of $\mathbb{A}^n$ at the coordinate functions)

$$\tilde{\mathbb{A}}^n = Bl_{x_1,\dots,x_n} \mathbb{A}^n \overset{\text{Lem}}{\subseteq} \underbrace{\{(x,y) \in \mathbb{A}^n \times \mathbb{P}^{n-1} : y_j x_i = y_i x_j \ \forall i,j\}}_{=: Z}$$

Claim This inclusion is an equality.
Pf $U_1 = \{(x,y) \in Z : y_1 \neq 0\}$ affine coord. $x_1, \dots, x_n, y_2, \dots, y_n$ setting $y_1 = 1$
$\Rightarrow$ Given $x_1, y_2, \dots, y_n$: equations $(*)$ say
$x_j = y_j \cdot x_1$ for $j = 2, \dots, n$
($\Rightarrow y_i x_j = x_1 y_i y_j = y_j x_i \ \forall i,j$) remaining eqns also satisfied.
$\Rightarrow \mathbb{A}^n \xrightarrow{\sim} U_1 \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$
$(x_1, y_2, \dots, y_n) \longmapsto ((x_1, x_1 y_2, \dots, x_1 y_n), (1 : y_2, \dots, y_n))$

Have seen:

- Z covered by irred. affine subsets $U_1, \dots, U_n \cong \mathbb{A}^n$
- $\dim U_i = n$ and $U_i \cap U_j \ni ((1,\dots,1),(1:\dots:1))$ non-empty

$\Rightarrow$ Z irred. of dimension n.

$$\begin{matrix} Z \\ \cup\!\shortmid \\ \mathrm{Bl}_{x_1,\dots,x_n}\mathbb{A}^n \end{matrix}$$

$\to$ irreducible as closure of $U = \mathbb{A}^n \setminus V(x_1,\dots,x_n)$

$\to$ $\dim = \dim(U) = n$ (birat'l invariant)

$\Rightarrow$ $\mathrm{Bl}_{x_1,\dots,x_n}\mathbb{A}^n = Z$ $\#$

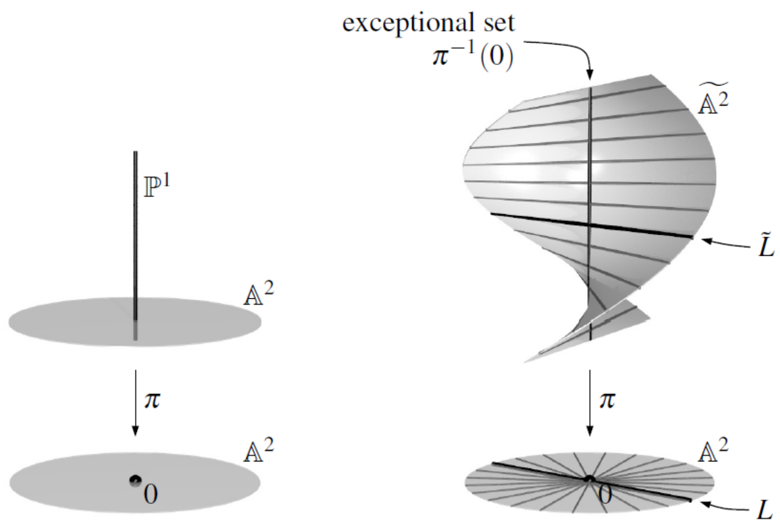
Blow-up morphism

equation $x_i y_j = x_j y_i$ satisfied $\forall y$

$$\begin{tikzcd} U = \mathbb{A}^n\setminus\{0\} \subseteq \arrow[d, "\sim"] & \tilde{\mathbb{A}}^n \arrow[d, "\pi"] & \supseteq \{(0:y) : y \in \mathbb{P}^{n-1}\} \cong \mathbb{P}^{n-1} \arrow[d] \\ U = \mathbb{A}^n\setminus\{0\} \overset{\text{open}}{\subseteq} & \mathbb{A}^n & \ni \{0\} \end{tikzcd}$$

exceptional set $\pi^{-1}(\{0\})$

$\rightsquigarrow$ blow-up leaves $\mathbb{A}^n\setminus\{0\}$ unchanged, replaces 0 by $\mathbb{P}^{n-1}$



Wrong picture Correct picture

Strict transform $\tilde{L}$ of line $L = \{(\lambda v_1, \dots, \lambda v_n) : \lambda \in K\}$

$=$ closure of $\{((\lambda v_1, \dots, \lambda v_n), (v_1 : \dots : v_n)) : \lambda \neq 0\}$

$= \{((\lambda v_1, \dots, \lambda v_n), (v_1 : \dots : v_n)) : \lambda \in K\}$

$\Rightarrow$ $\pi^{-1}(0)$ parameterizes directions through origin.
$\cong \mathbb{P}^{n-1}$

Generalized blow-ups

Have seen $Bl_{f_1,\dots,f_r}X$ only changes X along $V(f_1,\dots,f_r)$

Lem X affine, $f_1,\dots,f_r \in A(X)$
$\Rightarrow Bl_{f_1,\dots,f_r}X$ only depends on ideal $\mathcal{J} = \langle f_1,\dots,f_r\rangle \trianglelefteq A(X)$
More precisely: for $f_1',\dots,f_s' \in A(X)$ with $\mathcal{J} = \langle f_1',\dots,f_s'\rangle$ there exists a diagram:

$$\begin{tikzcd}
\tilde{X} = Bl_{f_1,\dots,f_r}X \arrow[rr, "F", "\sim"'] \arrow[dr, "\pi"'] & & Bl_{f_1',\dots,f_s'}X = \tilde{X}' \arrow[dl, "\pi'"] \\
& X &
\end{tikzcd}$$

isomorphism

Pf By assumption:

$g_{ij}, h_{jk} \in A(X)$.

$$f_i = \sum_{j=1}^{s} g_{ij} f_j' \quad \text{and} \quad f_j' = \sum_{k=1}^{r} h_{jk} f_k \quad \text{in } A(X) \qquad (*)$$

Then we can set

$$F: \underset{\substack{\cap\\ X\times\mathbb{P}^{r-1}}}{\tilde{X}} \xrightarrow{\sim} \underset{\substack{\cap\\ X\times\mathbb{P}^{s-1}}}{\tilde{X}'}, \quad (x,y) \mapsto (x,y') = \left(x, \left(\sum h_{1k}(x)y_k : \dots : \sum h_{sk}(x)y_k\right)\right)$$

Check

- well-defined map to $X\times\mathbb{P}^{s-1}$: check $y' \neq 0$

$(*) \Rightarrow f_i = \sum_{j,k} g_{ij} h_{jk} f_k$ and $(y_1:\dots:y_r) = (f_1:\dots:f_r)$ on $U = X\setminus V(f_1,\dots,f_r) \subseteq \tilde{X} \subseteq X\times\mathbb{P}^{r-1}$

$\Rightarrow \quad y_i = \sum_{j,k} g_{ij} \underbrace{h_{jk} y_k}_{= y_j'}$ on U (and thus on $\bar{U} = \tilde{X}$)

So $y' = 0 \Rightarrow y = 0$ ↯ to $(x,y) \in X\times\mathbb{P}^{r-1}$.

- $\mathrm{im}(F) \subseteq \tilde{X}'$: for $(x,y) \in U$ have $(y_1:\dots:y_r) = (f_1(x):\dots:f_r(x))$
$\overset{(*)}{\Longrightarrow} (y_1':\dots:y_s') = (f_1'(x):\dots:f_s'(x)) \Rightarrow (x,y') \in U$ so $F(U) \subseteq \tilde{X}' \Rightarrow F(\underset{=\bar{U}}{\tilde{X}}) \subseteq \tilde{X}'$
- F isomorphism: symmetric construction (w/ g_{ij}) gives F^{-1}
- $\pi' \circ F = \pi$: obvious from formula. $\square$

Construction (generalized blow-ups)

(a) X affine, $J \trianglelefteq A(X)$ ideal with generat. $f_1,\dots,f_r$

$\Rightarrow Bl_J X := Bl_{f_1,\dots,f_r} X$ <u>blow-up of X at J</u> ← well-def. by Lem.

$Y \subseteq X$ closed subvariety

$\Rightarrow Bl_Y X := Bl_{I_X(Y)} X$ <u>blow-up of X at Y</u>

<u>Exa</u> $Bl_{x_1,\dots,x_n} \mathbb{A}^n$ above $=$ blow-up of $\mathbb{A}^n$ at origin.

<u>Note</u> $Bl_{\langle x^2, y\rangle} \mathbb{A}^2 \not\cong Bl_{\langle x,y\rangle}$ ⇝ Bl_J does <u>not</u> only dep. on $V_X(J)$

(b) X arbitrary variety, $Y \subseteq X$ closed subvariety

⇝ $\{U_i : i \in I\}$ affine cover of X

$\widetilde{U}_i = Bl_{U_i \cap Y} U_i$ can be glued along $\widetilde{U}_{ij} = Bl_{U_i \cap U_j \cap Y} \underbrace{U_i \cap U_j}_{\text{affine open}}$ [Ex. 5.23(a)]

$\Rightarrow Bl_Y X$ variety covered by $\widetilde{U}_i$

<u>Exercise</u> X prevariety, Y variety, $\pi : X \to Y$ morphism

Then: X is a variety $\Leftrightarrow$ For some [any] open cover $\{U_i : i \in I\}$ of Y: $\pi^{-1}(U_i)$ variety $\forall i$

<u>Hint</u> $U_i \times U_j = (U_i \cap U_j) \times (U_i \cap U_j) \cup (U_i \times U_j \setminus \Delta_{U_i \cap U_j})$

(c) Generalizing $Bl_J X$ to X arbitrary variety

⇝ $J =$ sheaf of ideals in $\mathcal{O}_X$ (not in this lecture)

X projective, $f_1,\dots,f_r \in S(X)$ homog. of same degree

⇝ $\Gamma = \{(x, (f_1(x) : \dots : f_r(x))) : x \in U\} \subseteq U \times \mathbb{P}^{r-1}$

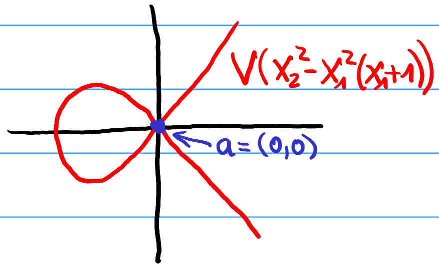
$U = X \setminus V_P(f_1,\dots,f_r)$

$\underbrace{Bl_{f_1,\dots,f_r} X}_{\text{projective}} = \overline{\Gamma} \subseteq \underbrace{X \times \mathbb{P}^{r-1}}_{\text{projective}}$ projective blow-up.

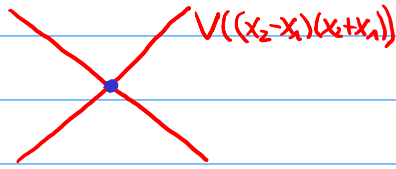
Tangent cones

One application of blow-ups: local study of varieties

Exa



Locally around a



↳ can be achieved via blow-up at a ?

Construction (Tangent cones)

X variety, $a \in X \rightsquigarrow \pi : \tilde{X} = Bl_a X \to X$ blow-up of a

Claim Exceptional set $\pi^{-1}(a)$ is a projective variety.

Pf $U \subseteq X$ affine neighbourhood ($U \subseteq \mathbb{A}^n$ aff. variety)

$\rightsquigarrow \tilde{X} = (X \setminus \{a\}) \cup_{U \setminus \{a\}} (Bl_a U)$ and $Bl_a U \subseteq U \times \mathbb{P}^{n-1}$

$\pi^{-1}(a) \subseteq \{a\} \times \mathbb{P}^{n-1}$ projective variety. □

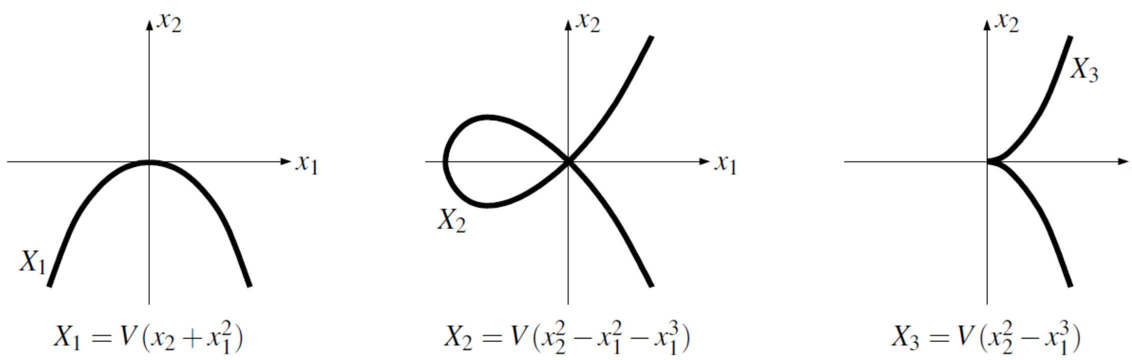
$C_a X := \operatorname{Cone}(\pi^{-1}(a))$ tangent cone of X at a

well-defined affine variety by [Lem 9.16]

For $a = 0 \in X \subseteq \mathbb{A}^n \rightsquigarrow C_a X \subseteq \operatorname{Cone}(\mathbb{P}^{n-1}) = \mathbb{A}^n$ ← closed subvar. in same aff. space $\mathbb{A}^n$ as X.

Examples

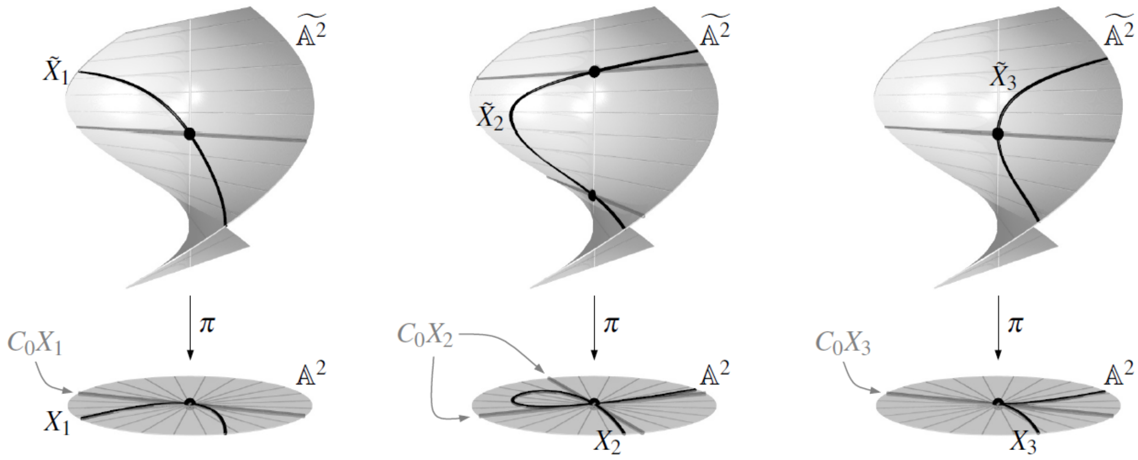
Consider the following curves $X_1, X_2, X_3 \subseteq \mathbb{A}^2$:



To compute $C_{(0,0)} X_i$:

$Bl_{(0,0)} X_i \subseteq Bl_{(0,0)} \mathbb{A}^2$ is strict transform $\tilde{X}_i$ of X_i

$\rightsquigarrow \pi^{-1}((0,0)) =$ intersect. of $\tilde{X}_i$ with except. set of $Bl_{(0,0)} \mathbb{A}^2 \to \mathbb{A}^2$.



Explicit calculation

$$\tilde{X}_2 \subseteq \Big\{((x_1,x_2),(y_1:y_2)) \in \mathbb{A}^2\times\mathbb{P}^1 : \overbrace{x_2^2-x_1^2-x_1^3=0}^{\text{equat. of } X},\ \overbrace{y_1x_2-y_2x_1=0}^{\text{extra equ. for proj. coord. } y_1,y_2}\Big\}$$

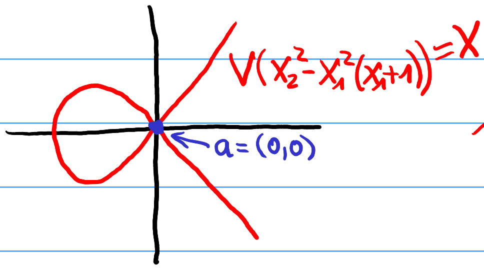
$\downarrow (y_1,y_2)=\lambda\cdot(x_1,x_2)$ when $(x_1,x_2)\neq(0,0)$

$\Rightarrow \lambda^2\cdot(x_2^2-x_1^2-x_1^3)=0$

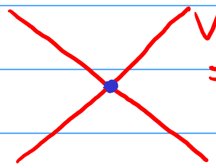
$\Rightarrow y_2^2-y_1^2-y_1^2x_1=0$ on $\tilde{X}_2\setminus\pi^{-1}((0,0))$ $\rightsquigarrow$ also on closure $\tilde{X}_2$

On $\pi^{-1}((0,0))$ $\overset{\text{set } x_1=x_2=0}{\rightsquigarrow}$ $y_2^2-y_1^2=(y_2-y_1)(y_2+y_1)=0$

$\Rightarrow (y_1:y_2)\in\{(1:1),(1:-1)\}=\pi^{-1}((0,0))$



Locally around a $\rightsquigarrow$



Note $x_2^2-x_1^2-x_1^3=0$ $\overset{\text{cone at }(0,0)}{\rightsquigarrow}$ $x_2^2-x_1^2=0$ ← take terms of lowest degree

Exercise (see [Ex. 9.22] for details & hints)

$J \trianglelefteq K[x_1,\dots,x_n]$ with $0\in X=V_a(J)$

$$\Rightarrow C_0X=V_a(f^{in} : f\in J)$$

↑ initial term = sum of monom. in f of lowest degree

$X=V_a(f)$ $\Rightarrow C_0X=V_a(f^{in})$

Dimensions of exceptional sets

In examples: $\underbrace{Bl_0 \mathbb{A}^n}_{\text{irred. dim } n}$ replaces $\{0\}$ by $\underbrace{\mathbb{P}^{n-1}}_{\text{dim } n-1}$

Pro (Dimension of the exceptional set)
X affine, $f_1,\dots,f_r \in A(X)$, $\pi: \tilde{X} = Bl_{f_1,\dots,f_r} X \to X$ blow-up
$\Rightarrow$ every irred. component of $E = \pi^{-1}(V(f_1,\dots,f_r))$ has codimension 1 in $\tilde{X}$
exceptional hypersurface (divisor)

Pf $\tilde{X} \subseteq X \times \mathbb{P}^{r-1}$ and $\mathbb{P}^{r-1} = U_1 \cup \dots \cup U_r$ affine cover
$\rightsquigarrow$ suffices to show claim on $V_i = \tilde{X} \cap X \times U_i$

$$\begin{array}{ccc} V_i & \subseteq & W_i := \{(x,y) \in X \times U_i : y_k f_j(x) \overset{(*)}{=} y_j f_k(x) \ \forall j,k\} \\ & {}_{\pi|_{V_i}}\searrow \quad \swarrow_{\pi_X} & \\ & X & \end{array}$$

Claim $\pi_X^{-1}(V_X(f_1,\dots,f_r)) = V_{W_i}(f_i)$ ← $(x,y) \mapsto f_i(x)$
Pf $(*)$ for $k=i$ (with $y_i = 1$):
$f_j(x) = y_j \cdot f_i(x) \rightsquigarrow (f_i(x) = 0 \Rightarrow f_j(x) = 0 \ \forall j)$ #

compute preim. via W_i
$\Rightarrow \pi_X^{-1}(V_X(f_1,\dots,f_r)) \cap V_i = V_{W_i}(f_i) \cap V_i = V_{V_i}(f_i)$

Claim 2 If $V_i \neq \emptyset \rightsquigarrow f_i \neq 0$ on V_i
Pf Otherwise $\pi(V_i) \subseteq V_X(f_1,\dots,f_r) \Rightarrow \pi(\underbrace{\tilde{X}}_{= \overline{V_i}}) \subseteq V_X(f_1,\dots,f_r) \rightsquigarrow U = \emptyset$
$\rightsquigarrow \tilde{X} = \overline{U} = \emptyset$ ↯

Krull's principal ideal thm [Pro 2.28 (c)] $\Rightarrow V_{V_i}(f)$ pure codim 1 in V_i □

As a consequence, we can also compute the dimension of the tangent cone to a variety X.

Cor (Dimension of tangent cones)

local dimens. of X at a

X variety, $a \in X \Rightarrow \dim C_a X = \operatorname{codim}_X \{a\}$

Exercise [Ex. 2.35]

X affine variety, $X = X_1 \cup \dots \cup X_r$ irred. decomp., $a \in X$

$\Rightarrow \operatorname{codim}_X \{a\} = \max \{ \dim X_i : a \in X_i \}$

Pf of Cor Shrinking X around a

$\rightsquigarrow$ can assume every irred. comp. of X meets a

& $X \subseteq \mathbb{A}^n$ affine (also: $X \neq \{a\}$, otherwise trivial).

$X = X_1 \cup \dots \cup X_m$ irred. decomposition $(\dim X_i \geq 1)$

$\overset{[\text{Pro}]}{\Longrightarrow}$ exceptional set of $\tilde{X}_i \to X_i$ pure dim. $\dim X_i - 1$

$\overset{[\text{Ex } 6.31]}{\Longrightarrow}$ $C_a X_i$ pure dim $\dim X_i$

$\rightsquigarrow$ maximum of these dimensions:

$\dim C_a X = \max_i \dim X_i \overset{[\text{Ex. } 2.35]}{=\!=} \operatorname{codim}_X \{a\}$.

Note we used that exc. set E of $\tilde{X}_i \to X_i$ not empty

Pf $U = X_i \setminus \{a\}$ then $\dim X_i \geq 1 \Rightarrow U \neq \emptyset \Rightarrow \boxed{\overline{U} = X_i}$ (+)

By def:

$$\begin{tikzcd} \tilde{X}_i = \overline{U} \subseteq X_i \times \mathbb{P}^{n-1} \arrow[d, "\tilde{\pi}"] \arrow[dr, phantom, ""] \\ X_i \end{tikzcd} \qquad \tilde{X}_i \xrightarrow{\pi} X_i$$

Have seen: $\tilde{\pi}$ closed

($\mathbb{P}^{n-1}$ complete)

$\Rightarrow \pi(\tilde{X}_i)$ closed, contains U

$\overset{(+)}{\Rightarrow} \pi(\tilde{X}_i) = X_i \Rightarrow E = \pi^{-1}(a)$ not empty. $\square$

Blowing up to remove indeterminacies of rational maps

Second application blow up domain of rat'l map $X \dashrightarrow Y$ to get morphism $\tilde{X} \to Y$

Exa X affine variety, $f_1, \dots, f_r \in A(X)$

$\rightsquigarrow$ $X \overset{f}{\dashrightarrow} \mathbb{P}^{r-1}$, $x \mapsto (f_1(x) : \dots : f_r(x))$ defined on $U = X \setminus V_X(f_1 \dots f_r)$

In general: cannot extend further (e.g. $X = \mathbb{A}^r$, $f_i = x_i$)

However:

X irred., U dense $\Rightarrow$ π birat'l.

$$\begin{tikzcd} \tilde{X} = Bl_{f_1,\dots,f_r} X \subseteq X \times \mathbb{P}^{r-1} \arrow[d, "\pi"'] \arrow[dr, "\pi_2"] & \\ X \arrow[r, dashed, "f"] & \mathbb{P}^{r-1} \end{tikzcd}$$

get $\tilde{X} \longrightarrow \mathbb{P}^{r-1}$

Exa Know that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birational.
Can we write down birat'l map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ & resolve indeterm.?

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow[\text{Segre emb.}]{\sim} X = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3 : x_0 x_3 = x_1 x_2\} \subseteq \mathbb{P}^3$$

$$X \overset{f}{\dashrightarrow} \mathbb{P}^2, \quad \mapsto (x_0 : x_1 : x_2)$$

$\leftsquigarrow$ projection from $a = (0:0:0:1)$.

Claim

$$\begin{tikzcd} Bl_a X \arrow[r, "\hat{f}", "\sim"'] \arrow[d] & Bl_{\{b,c\}} \mathbb{P}^2 \arrow[d] \\ \mathbb{P}^1 \times \mathbb{P}^1 = X \arrow[r, dashed, "f"] & \mathbb{P}^2 \end{tikzcd}$$

$b = (0:1:0)$
$c = (0:0:1)$

$\rightsquigarrow$ made f into isomorph. by blowing up $\begin{cases} \text{one pt in } \mathbb{P}^1 \times \mathbb{P}^1 \\ \text{two pts in } \mathbb{P}^2. \end{cases}$
[blowing up b, c resolves indet. of f^{-1}]

Two proofs

- explicit calculation with coordinates [Lem. 9.27]
- geometric interpretation via project. from a [Rmk 9.28].